Mechanisms of ergodic torus destruction and appearance of strange nonchaotic attractors

V. S. Anishchenko,* T. E. Vadivasova,† O. Sosnovtseva‡
Physics Department, Saratov State University, Astrakhanskaya st. 83, Saratov, 410071, Russia
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We present the results of the computer simulation of the dynamics of an invertible two-dimensional ring map with quasiperiodic excitation. The bifurcation diagram of the system on the parameter plane has been constructed. We analyze the mechanisms of the destruction of quasiperiodic regimes and the role of strange nonchaotic attractors (SNA) in this process. Two mechanisms of SNA appearance are discussed. We verify the existence of SNA via bifurcational analysis of the approximating sets of the attractor.

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I. INTRODUCTION

Recently interest in the study of self-oscillatory systems with quasiperiodic external forcing (i.e., the forcing by the signal with two incommensurate frequencies) has appeared. The simplest attractor in such systems is an ergodic torus T^2 (in terms of map, it corresponds to a closed invariant curve). Quasiperiodic motion on T^2 can undergo a finite number of period doubling bifurcations and can be destroyed as a result of torus fractalization and the appearance of chaos [1–4]. According to [3] the so-called fractal torus exists only at the chaos boundary. Besides the quasiperiodic and chaotic attractors, the strange nonchaotic attractor (SNA) can exist in systems with quasiperiodic excitation over the sets of positive measure in parameter space [5–11]. SNA is characterized by the fractal geometric structure but the trajectory divergence on it is absent. The motion on SNA has a singular continuous spectrum and decreasing autocorrelation function [6–12]. The capacity dimension of SNA differs greatly from its information dimension [13,14].

The mechanisms of SNA formation still are not understood completely. One of the possible ways for the SNA to emerge is described in [15]. It is as follows: the two bands of a doubled torus are separated by an unstable torus (in the case of a one-dimensional map with the quasiperiodic forcing) or by a stable manifold of a saddle torus (in the general case). A variation of the bifurcation parameter leads to the doubled torus becoming wrinkled and touching the separatrix. The separatrix is destroyed and the two bands of the torus merge. At this moment the doubled torus transforms into the fractal set while the largest nonzero Lyapunov exponent of the motion remains negative. But this mechanism of SNA formation cannot be universal because SNA is observed also without preliminary torus doubling or band-merging crisis.

The role of SNA in the transition from the quasiperiodic regime to chaos is not still absolutely clear. Two routes to chaos from ergodic two-dimensional torus were proposed. One of them described in [16] is as follows:

two-frequency quasiperiodicity

→three-frequency quasiperiodicity→SNA→chaos,

and the other one considered in [17] is

two-frequency quasiperiodicity

→SNA→three-frequency quasiperiodicity→chaos.

But we have to note that three-frequency quasiperiodicity is not a necessary stage on the route to chaos or to the nonchaotic strange attractor. The transition from T^2 to SNA and chaos can be observed in the systems where the three-frequency quasiperiodic motion cannot exist, for example, in the quasiperiodically driven logistic map [15].

Thus, the following questions remain: What is the connection between the appearance of SNA and the process of the destruction of the two-dimensional ergodic torus? Is the fractal torus of Kaneko [3] the same object as SNA? How is the transition from strange nonchaotic attractor to strange chaotic attractor accomplished? To answer these questions the mechanisms of invariant curve destruction in the map with the quasiperiodic excitation are analyzed in this paper in detail. In Sec. II we introduce the quasiperiodically forced ring map and give a survey of the behavior observed in this model. Section III is devoted to a detailed analysis of the transition to chaos. Finally, the results are summarized in Sec. IV.

II. BASIC MODEL

The quasiperiodically forced ring map has the following form:

$$\begin{aligned} X_{n+1}&=X_n+\Omega-K/2\pi\sin(2\pi X_n)+\gamma Y_n \\ &\quad +A\cos(2\pi Z_n), \quad \mod 1 \\ Y_{n+1}&=\gamma Y_n-K/2\pi\sin(2\pi X_n), \qquad (1) \\ Z_{n+1}&=Z_n+\omega, \quad \mod 1, \end{aligned}$$

where Ω, K, γ are the internal parameters of the system. K is the parameter of nonlinearity, Ω is the parameter controlling the natural winding number Q of the system, γ characterizes the dissipation degree. When $\gamma=0.01$ the dynamics

*Electronic address: wadim@chaos.ssu.runnet.ru
†Electronic address: tanya@chaos.ssu.runnet.ru
‡Electronic address: olga@chaos.ssu.runnet.ru; to whom correspondence should be addressed.

of the map (1) is practically the same as the dynamics of the well-known circle map with the quasiperiodic excitation:

$$X_{n+1}=X_n+\Omega-K/2\pi\sin(2\pi X_n)+A\cos(2\pi Z_n), \quad \text{mod } 1,$$

$$Z_{n+1}=Z_n+\omega, \quad \text{mod } 1. \tag{2}$$

In contrast to the system (2) the system (1) is invertible and, therefore, it corresponds to the Poincare section of the flow. The parameters A and ω are the amplitude and the frequency of the external forcing, respectively. The forcing is quasiperiodic because of the presence of two incommensurate frequencies, here they are 1 and ω.

Forcing frequency ω is irrational and chosen to be equal to golden mean $\omega=(\sqrt{5}-1)/2$. It depends on Ω whether the three-frequency quasiperiodic motion T^3 or the two-frequency quasiperiodic motion qT^2 is observed in the system (1). The attractor qT^2 corresponds to the synchronization regions with the internal winding number $Q=p/q$, where p and q are the integer numbers. The number q defines the number of the invariant curve bands. The ergodic motion on qT^2 in the synchronization region is the rough regime because the ω is fixed and has an irrational value. When the parameters A and K are varied the destruction of the invariant curve qT^2 and the transition to chaos can be observed.

III. MECHANISMS OF INVARIANT CURVE DESTRUCTION AND THE APPEARANCE OF SNA

Let us consider the destruction of the invariant curve qT^2. We keep $\Omega=0.5$ (when $Q=0.5$) and $A=0.4$ and increase the parameter K. At the beginning the smooth curve has two bands. The phase point visits each of them exactly in one iteration of the map. When the parameter K is increased, the distortion of the shape of the invariant curve takes place. For $K=K_1\simeq1.030$ two bands of the invariant curve qT^2 merge. The regularity of the phase point's visiting two bands of the attractor is broken. The attractor arising after the merging crisis is like the fractal set. Meanwhile, the largest nonzero Lyapunov exponent remains negative during and after the crisis as noted in [15].

To give reliable evidence of the existence of SNA for $K>K_1$ the rational approximation method proposed in [18] was used. This method is as follows: The irrational value of $\omega=(\sqrt{5}-1)/2$ is replaced by the rational approximation $\omega_k=F_k/F_{k+1}$, where $F_k=1,1,2,3,5,\ldots$ are the Fibonacci numbers. In this case the limit set is the cycle with the period F_{k+1}. Equation (1) is a periodically [with period $F(k+1)$] forced map and it can have an attractor that depends on the initial phase Z_0. The dependence $X(Z_0)$ represents the so-called approximating attracting set where X is the coordinate of limit set points for each Z_0 from the interval $[0;1/F_{k+1}]$. As shown in [18] if the approximating set is nonsmooth for sufficiently large k, or the maximum derivative $\max\{dX_j/dZ_0\}$, where j belongs to all bands of the attracting set, grows indefinitely with k then for the irrational value of ω [$\omega=\lim_{k\to\infty}(F_k/F_{k+1})$] the system has an attractor that is not piecewise differentiable and, hence, it is strange (chaotic or nonchaotic) [5].

Figures 1 and 2 show nonstrange and strange nonchaotic

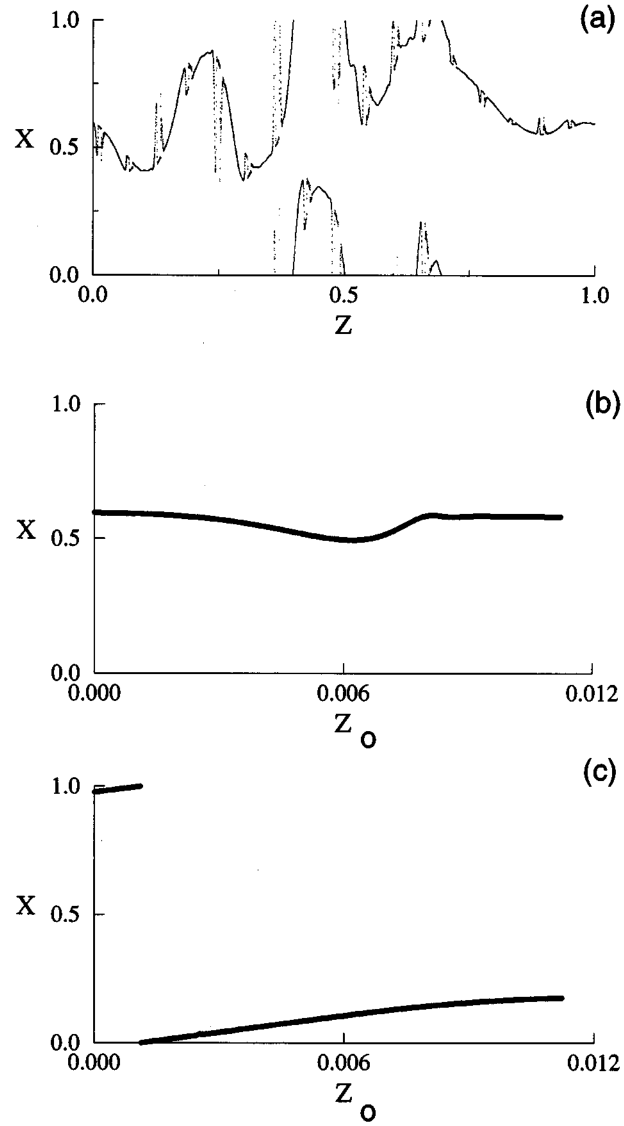

FIG. 1. (a) ZX projections of the attractor and (b),(c) its approximating sets with $\omega=55/89$ for $\Omega=0.5$, $A=0.4$, and $K=1.029$. The phase projections of the attractor were obtained for each second iteration and corresponding approximating sets were constructed for even (b) and odd (c) iterations.

attractors, respectively. To distinguish the moment of the appearance of SNA via the band merging crisis we represent the phase portraits only for each second iteration. The attracting set for the rational approximation $\omega=55/89$ is constructed for even and odd iteration numbers. For $K=1.029$ the invariant curve [Fig. 1(a)] as well as the approximating set [Figs. 1(b), 1(c)] have two bands. Because the attracting set of the rational approximation is smooth, there exists the quasiperiodic motion on $2T^2$ in the system (1) for this parameter value. For $K=1.031$ the phase portrait of the attractor has the same shape [Fig. 2(a)] as would be obtained for each iteration. This fact gives evidence that the band merging crisis has taken place. In this case, the approximating set has an interesting structure: two bands are sewed together over the same parts; i.e., there is a common set of points for even and odd iteration numbers [Figs. 2(b), 2(c)]. The smoothness of the approximating set is broken at the ends of these

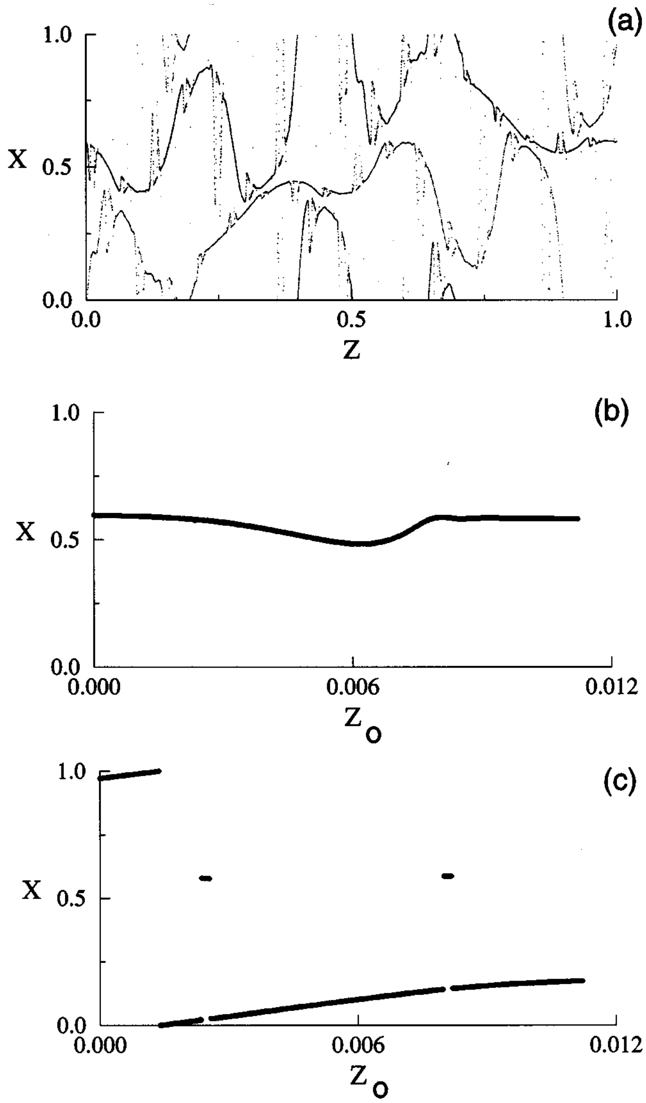

FIG. 2. (a) ZX projections of the attractor and (b),(c) its approximating sets with $\omega=55/89$ for $\Omega=0.5$, $A=0.4$, and $K=1.031$. The phase projections of the attractor were obtained for each second iteration and corresponding approximating sets were constructed for even (b) and odd (c) iterations.

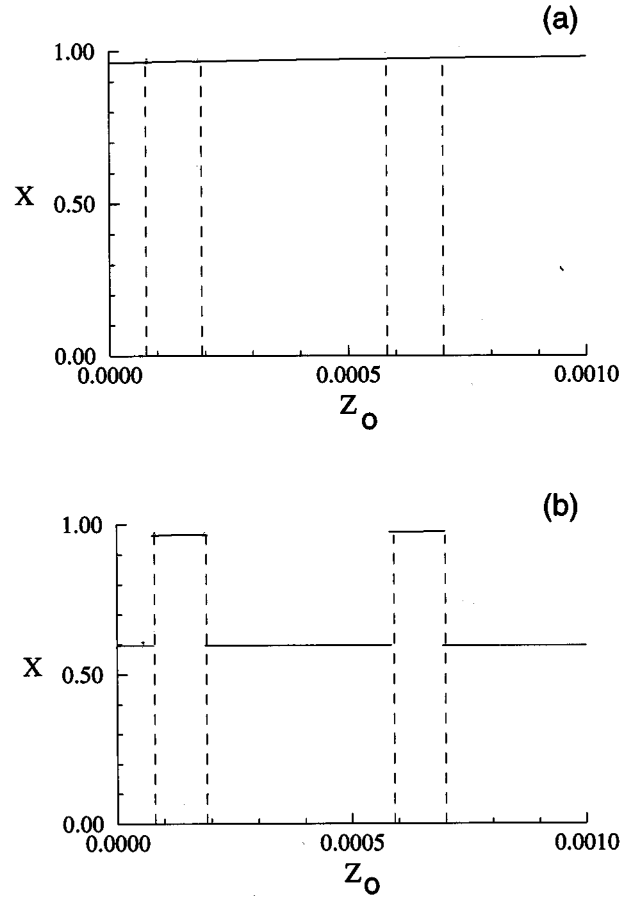

FIG. 3. Two bands of the approximating attracting set with $\omega=610/987$ for the attractor for $\Omega=0.5$, $A=0.4$, $K=1.031$.

"sewed" parts and the fractal attractor arises in the system (1). Because the largest nonzero Lyapunov exponent λ_1 remains negative, it is a strange nonchaotic attractor. Further increasing the parameter K leads to the complete merging of two bands of the attractor. The approximating set becomes everywhere unified (for even or odd iterations); i.e., for each Z_0 there exists only one value of X.

To confirm the nonsmooth properties of the attractor after band merging, let us construct the approximating set for $\omega=610/987$. The approximating sets of the attractor with two bands for $K=1.031$ are given in Figs. 3(a), 3(b). It is clearly seen that the character of the approximating set of the attractor is the same as for $\omega=55/89$ [Figs. 2(b), 2(c)]. In Fig. 3 we can distinguish the parts of two bands that are sewed. They are marked by the dashed lines.

Let us consider the point of the attracting set for the rational approximation $\omega_k=F_k/F_{k+1}$ corresponding to the fixed value of Z_0. It is the fixed point of a certain map that is obtained from the map (1) applied for $k+1$ times. This map depends not only on the parameters of the map (1) but also on the initial phase Z_0. Taking into account a very insignificant dependence of variable X on variable Y for a small value of the parameter γ (the case of strong dissipation) in the system (1) the map may be represented in the following one-dimensional form:

$$X_{n+1}=G_k(X_n,Z_0). \tag{3}$$

The behavior of the points of the approximating set is determined by the map (3). During the destruction of the invariant curve the dependence on Z_0 becomes essential. The character of the return function (Fig. 4) corresponding to the approximating set shown in Fig. 3 is quite different for two values of Z_0. In the case of $Z_0=0$ the map (3) has the stable limit cycle with period 2 [Fig. 4(a)], while in the case of $Z_0=0.00014$ it has the stable fixed point [Fig. 4(b)].

The method of rational approximation for $\omega=610/987$ has shown that for parameter values $\Omega=0.5$, $A=0.4$ the strange nonchaotic attractor that appears via the band merging exists only for $K\in[K_1,K_2]$. For $K>K_2$ the attractor becomes again regular and represents the piecewise differentiable manifold. This manifold is not homeomorphic to the smooth invariant curve because it has the points of discontinuity. With the further increase of K the chaotic attractor

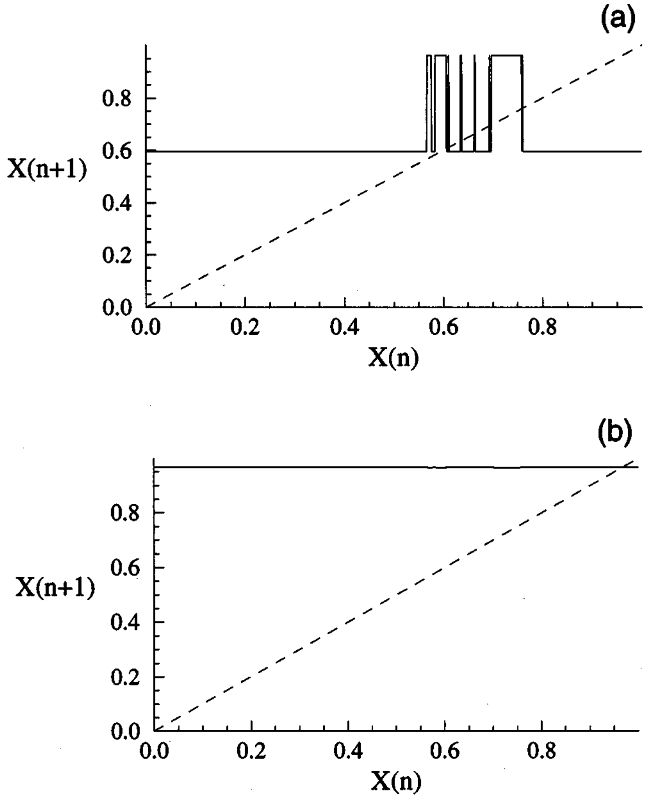

FIG. 4. Map $G_k(X)$ for $\omega=610/987$, $\Omega=0.5$, $A=0.4$, $K=1.031$, and $Z_0=0$ (a); $Z_0=0.00014$ (b).

emerges (for $K\geqslant 1.520$ the largest nonzero Lyapunov exponent becomes positive). Probably, the arising of SNA for the second time precedes chaos.

Thus, we have considered one of the possible bifurcational sequences for the transition from the smooth ergodic torus to chaos: the destruction of the torus via the band merging crisis and development of SNA; degradation of SNA to a piecewise differentiable manifold with a finite number of points of discontinuity; the fractalization of the attractor for the second time and the appearance of the chaotic attractor. However, SNA arisen via the band merging is not obliged to degenerate to the regular attractor and can develop immediately to the chaotic attractor. Such a scenario takes place, for example, for $\Omega=0.5$, $A=0.15$ with the increase of the parameter K. The destruction of the invariant curve and the appearance of SNA can also be the result of merging of two different tori coexisting in the phase space of the system. The crisis of the torus in such a case is also connected with the separatrix destruction and it is similar to the band merging crisis described above. In the system (1) the merging crisis of two different tori is observed, for example, for $\Omega=0.5$, $A=0.3$. For $1.04<K<1.05$ two invariant curves $2T_1^2$ and $2T_2^2$ possessing the mutual symmetry ($X\rightarrow 1-X$; $Z\rightarrow Z+0.5$) merge together. The arisen fractal set gradually transforms to chaos.

Figure 5 shows the fragment of the bifurcation diagram of the system (1) on the parameter plane (A,K) for $\Omega=0.5$. The following bifurcation lines are shown on the diagram: l_1 is the line of the band merging of the invariant curve; l_2 is the line of merging of two different invariant curves; l_3 is the

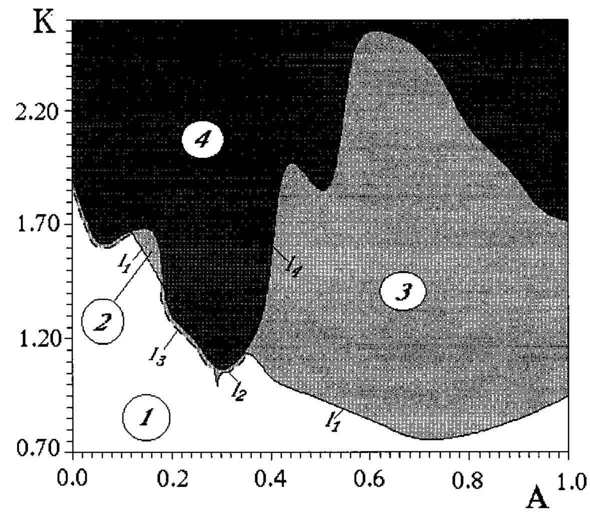

FIG. 5. The fragment of the bifurcation diagram on the A-K parameter plane for $\Omega=0.5$.

chaos boundary (it corresponds to the appearance of the positive Lyapunov exponent). The regions with different dynamics are marked by the following numbers: 1 is the region where the smooth invariant curve $2T^2$ exists; 2 is the region where either the SNA or piecewise differentiable attractor is observed; 3 is the chaotic region. The line of the band merging l_1 crosses the chaos boundary l_3. It means that the band merging crisis does not always precede the appearance of chaos. How is the invariant curve destroyed in this case? Evidently it is destroyed as a result of the gradual fractalization process described in [3]. In Fig. 5 the curve l_4 is the boundary of the appearance of SNA via gradual fractalization.

Let us consider in detail the process of gradual fractalization of the ergodic invariant curve. It is convenient to choose $\Omega=0.1$ because the invariant curve has one band. The amplitude of excitation is fixed ($A=0.3$) and the nonlinearity parameter K is changed. With the increase of K the distortions of the invariant curve are observed. The so-called torus oscillations take place [3]. The scale of these oscillations becomes smaller and smaller until the invariant curve is destroyed. For $K=K_{\rm cr}\simeq 2.0295$ the largest nonzero Lyapunov exponent λ_1 becomes positive. The trajectory divergence appears. Does the fractal set arise at the same moment or does it happen earlier? In other words, does the strange nonchaotic attractor precede chaos? According to [4] the fractal set emerges only on the chaos boundary and, therefore, SNA exists for only one value of K (for the fixed value of A). However, in spite of the results of the precise numerical investigations given in [4] such a conclusion causes doubt. Figure 6 shows ZX projections of attractors and their approximating sets for the rational approximation $\omega=610/987$ and $K=2.029$ (a), $K=2.050$ (b). In case (a) the largest nonzero Lyapunov exponent is negative and equals $\lambda_1=-0.011489$. Case (b) corresponds to the chaotic dynamics with $\lambda_1=+0.048767$. The approximating sets in cases (a) and (b) are not apparently everywhere differentiable manifolds and this confirms the nonsmooth properties of the attracting set.

Let us consider the largest Lyapunov exponent $\lambda_1^{(k)}$,

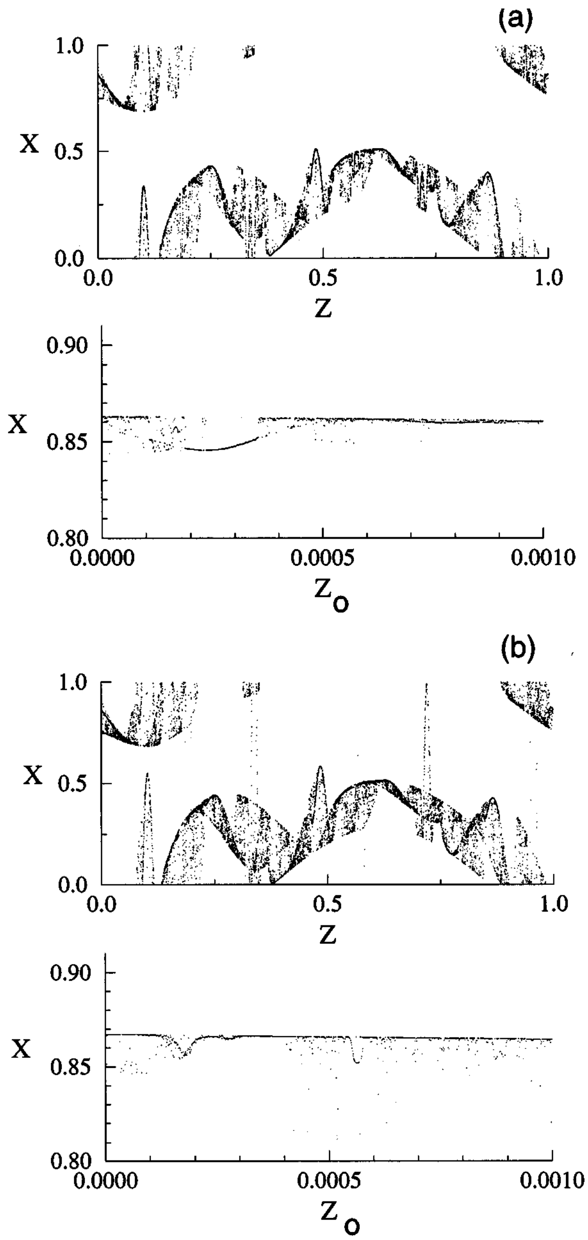

FIG. 6. XZ projections of the attractors and their approximating sets with $\omega=610/987$ for $\Omega=0.1$, $A=0.3$, and $K=2.029$ (a); $K=2.050$ (b).

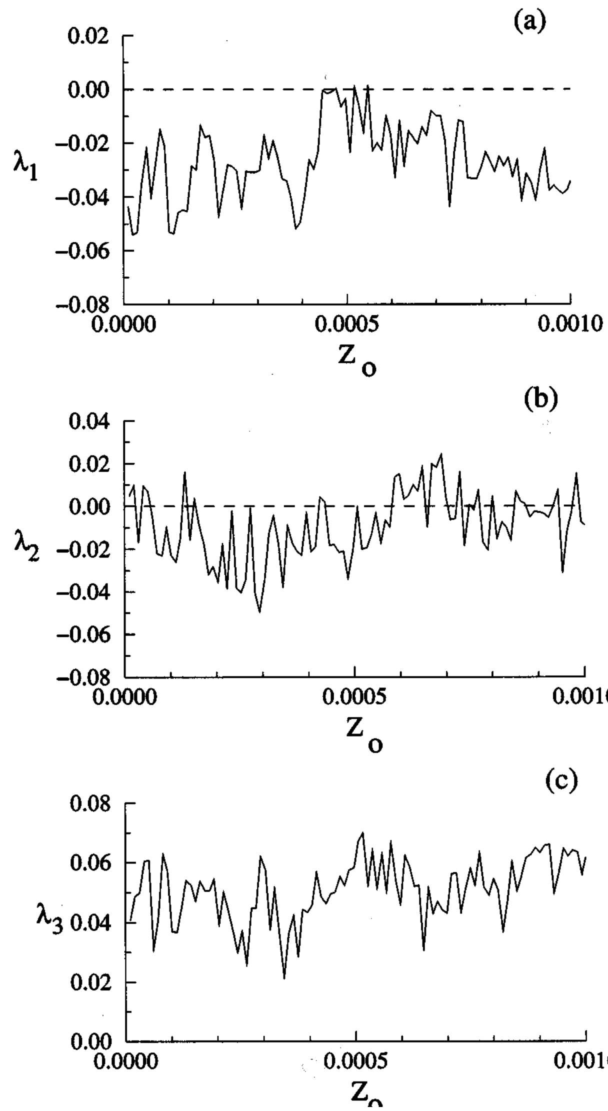

FIG. 7. The largest Lyapunov exponent of the trajectory as function of initial phase [$\omega=610/987$, $\Omega=0.1$, $A=0.4$, and $K=2.029$ (a); $K=2.05$ (b)].

which characterizes the stability of motion corresponding to the rational approximation of $\omega_k=F_k/F_{k+1}$. The calculations (they were performed for $\omega=610/987$ and 987/1597) for $K=2.020$ and 2.029 show that the Lyapunov exponent $\lambda_1^{(k)}$ can be both positive and negative depending on the choice of the initial phase Z_0. The dependence of $\lambda_1^{(k)}$ on Z_0 obtained for the rational approximation $\omega=610/987$ and for the three values of nonlinearity parameter $K=2.020$, $K=2.029$, and $K=2.050$ are presented in Figs. 7(a)–7(c), respectively. In cases (a) and (b) when SNA exists the Lyapunov exponent $\lambda_1^{(k)}$ changes its sign depending on Z_0. Hence, for some values of Z_0 the map $G_k(X)$ has chaotic dynamics. The number of points of Z_0 for which $\lambda_1^{(k)}>0$ increases as K approaches the critical value K_{cr} corresponding to the transition to chaos. The Lyapunov exponent λ_1 of the ergodic motion is like the result of averaging the Lyapunov exponent $\lambda_1^{(k)}$ over all values of $Z\in[0,1/F_{k+1}]$. While $\lambda_1^{(k)}<0$ for most values of Z_0, the Lyapunov exponent λ_1 of ergodic motion remains negative. When the set of Z_0 for which $\lambda_1^{(k)}(Z_0)>0$ begins to prevail the ergodic motion becomes chaotic. Figure 7(c) corresponds to the chaotic attractor in the case when the condition $\lambda_1^{(k)}(Z_0)>0$ is fulfilled for all Z_0.

Thus, in cases (a) and (b) the local divergence of trajectories on the attractor takes place though the global trajectory divergence is absent. The attractor in such a case can be neither chaotic nor regular. Hence, we can propose that the attractor for $K\geqslant 2.020$ is strange and the fractalization precedes chaos. The transition to chaos is realized gradually via the appearance of the local trajectory divergence.

The Lyapunov exponent $\lambda_1^{(k)}$ mentioned above has a similar sense to the local Lyapunov exponent introduced in [18] and the transient Lyapunov exponent [19]. In fact, for the calculation of the Lyapunov exponent λ_1 of ergodic motion during the finite time the result will be close to $\lambda_1^{(k)}$ because the points of the attractor received by iteration during some time will lie near to the points of the approximating set. As shown in [18] the presence of the positive local Lyapunov exponent is the sufficient condition for the existence of the fractal set (though it is not necessary).

IV. CONCLUSIONS

The fulfilled investigations allow one to conclude that there are two universal routes to the destruction of an ergodic two-dimensional torus:

(1) The crisis of the torus (merging of either two bands of the doubled torus or of two different tori). The torus crisis results in the instantaneous appearance of a strange nonchaotic attractor. Further, SNA either transforms to a chaotic attractor or degenerates to the regular attractor, which is not homeomorfic to the torus. (2) The gradual fractalization of the torus leads to the appearance of SNA, which further transforms to chaos.

Thus, the destruction of the ergodic torus always results in the appearance of a fractal set without trajectory divergence, which can then develop to chaos. Therefore, the regime of SNA is typical for systems with quasiperiodic excitation. The chaotization of motion on the strange attractor is accomplished gradually, beginning with the appearance of the local trajectory divergence, which becomes prevalent over all attractors with the transition to chaos. The results obtained for other model systems confirm these conclusions [15].

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